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2000 J. Phys. A: Math. Gen. 33 L375

(http://iopscience.iop.org/0305-4470/33/41/103)

J. Phys. A: Math. Gen. **33** (2000) L375–L380. Printed in the UK PII: S0305-4470(00)16112-8

LETTER TO THE EDITOR

Another addition theorem for the q-exponential function

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Received 4 August 2000

Abstract. An 'addition' formula with respect to a variable and parameter is established for the basic exponential function on a q-quadratic grid.

A basic exponential function on a q-quadratic grid can be introduced as

$$\mathcal{E}_q(x, y; \alpha) = \frac{(\alpha^2; q^2)_\infty}{(q\alpha^2; q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2/4}\alpha^n}{(q; q)_n} \mathrm{e}^{-\mathrm{i}n\varphi}(-q^{(1-n)/2}\mathrm{e}^{\mathrm{i}\theta+\mathrm{i}\varphi}, -q^{(1-n)/2}\mathrm{e}^{\mathrm{i}\varphi-\mathrm{i}\theta}; q)_n \tag{1}$$

where $x = \cos\theta$ and $y = \cos\varphi$ and $|\alpha| < 1$ (see [1–5, 10–12, 14] and see [15] for more details including representation in terms of basic hypergeometric series and analytic continuation in a larger domain).

We use the standard notations for the basic hypergeometric series

$${}_r\varphi_s\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; q, t\right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, b_2, \ldots, b_s; q)_n}((-1)^n q^{n(n-1)/2})^{1+s-r} t^n \tag{2}$$

and for the q-shifted factorials

$$(a; q)_n := \prod_{k=0}^{n-1}(1 - aq^k) \qquad (a_1, a_2, \ldots, a_r; q)_n := \prod_{k=1}^{r}(a_k; q)_n \tag{3}$$

$$(a; q)_\infty := \lim_{n\to\infty}(a; q)_n \qquad (a_1, a_2, \ldots, a_r; q)_\infty := \prod_{k=1}^{r}(a_k; q)_\infty \tag{4}$$

provided $|q| < 1$. See [6] for an excellent account of the theory of basic hypergeometric series.

Function $\mathcal{E}_q(x, y; \alpha)$ is an analogue of $\exp[\alpha(x + y)]$,

$$\lim_{q\to 1^-} \mathcal{E}_q(x, y; (1 - q)\alpha/2) = \exp[\alpha(x + y)]. \tag{5}$$

We also introduce

$$\mathcal{E}_q(x; \alpha) = \mathcal{E}_q(x, 0; \alpha) \tag{6}$$

as the q-analogue of $\exp(\alpha x)$. The following properties hold:

$$\mathcal{E}_q(0, 0; \alpha) = \mathcal{E}_q(0; \alpha) = 1 \qquad \mathcal{E}_q(x, y; \alpha) = \mathcal{E}_q(y, x; \alpha). \tag{7}$$

A commutative q-analogue of the addition theorem $\exp[\alpha(x + y)] = \exp(\alpha x)\exp(\alpha y)$ has been established by the author in [14].

0305-4470/00/410375+06$30.00

Theorem 1.

$$\mathcal{E}_q(x, y; \alpha) = \mathcal{E}_q(x; \alpha)\mathcal{E}_q(y; \alpha). \tag{8}$$

This formula has attracted some attention and different proofs of this relation were given in [4, 11], and [14].

Although $\mathcal{E}_q(x; \alpha)$ is an analogue of $\exp(\alpha x)$, the function $\mathcal{E}_q(x; \alpha)$ is not symmetric in x and α, so one would expect $\mathcal{E}_q(x; \alpha)$ to have two different addition theorems. Equation (8) gives the addition theorem in the variable x. Ismail and Stanton [11] have recently found the following expansion formula:

$$\begin{aligned}(q\alpha^2, q\beta^2; q^2)_\infty\, \mathcal{E}_q(x; \alpha)\, \mathcal{E}_q(x; \beta)\\ = \sum_{n=0}^{\infty} q^{n^2/4}\alpha^n\, H_n(x|q)(-\alpha\beta q^{(n+1)/2}; q)_\infty \frac{(-q^{(1-n)/2}\beta/\alpha; q)_n}{(q; q)_n}\end{aligned} \tag{9}$$

where $H_n(x|q)$ are the continuous q-Hermite polynomials,

$$H_n(\cos\theta|q) = \sum_{k=0}^{n} \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} \mathrm{e}^{\mathrm{i}(n-2k)\theta} \tag{10}$$

(see, e.g., [6] and [13]). Expansion (9) and its extensions found in [11] are important contributions in the growing area of q-series (see, e.g., [15] for applications of these expansions to the theory of basic Fourier series [1]). Ismail and Stanton [11] have called (9) the addition theorem in the variable α because it becomes $\exp(\alpha x)\exp(\beta x) = \exp[(\alpha + \beta)x]$ in the limit $q \to 1^-$.

In this Letter the author would like to present another version of the 'addition' formula with respect to both variables x and α, which extends theorem 1.

Theorem 2.

$$\begin{aligned}\mathcal{E}_q(\cos\theta; \alpha)\mathcal{E}_q(\cos\varphi; \beta) = \frac{(\beta^2; q^2)_\infty}{(q\alpha^2; q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2/4}\beta^n}{(q; q)_n}\\ \times \mathrm{e}^{-\mathrm{i}n\varphi}(-q^{(1-n)/2}\mathrm{e}^{\mathrm{i}\theta+\mathrm{i}\varphi}\alpha/\beta, -q^{(1-n)/2}\mathrm{e}^{\mathrm{i}\varphi-\mathrm{i}\theta}\alpha/\beta; q)_n\\ \times {}_2\varphi_2\left(\begin{matrix} q^{-n}, \alpha^2/\beta^2 \\ -q^{(1-n)/2}\mathrm{e}^{\mathrm{i}\theta+\mathrm{i}\varphi}\alpha/\beta, -q^{(1-n)/2}\mathrm{e}^{\mathrm{i}\varphi-\mathrm{i}\theta}\alpha/\beta \end{matrix}; q, q\mathrm{e}^{2\mathrm{i}\varphi}\right).\end{aligned} \tag{11}$$

This formula can be thought of as a general analogue of $\exp(\alpha x)\exp(\beta y) = \exp(\alpha x + \beta y)$. Clearly, our theorem 2 gives the addition formula (8) when $\beta = \alpha$. It is natural to denote the right-hand side of (11) as $\mathcal{E}_q(x, y; \alpha, \beta)$, then

$$\mathcal{E}_q(x; \alpha)\mathcal{E}_q(y; \beta) = \mathcal{E}_q(x, y; \alpha, \beta). \tag{12}$$

The case of the addition theorem in the variable α, raised by Ismail and Stanton [11], arises when $y = x$. Theorem 1 simplifies the product of two single series to a similar single series, while theorem 2 allows us to factor the double series into a product of two single series.

Proof. Our proof of (11) is based on the connection relation (10.2) of [1], which we rewrite here as

$$\frac{(q\alpha^2; q^2)_\infty}{(q\beta^2; q^2)_\infty}\mathcal{E}_q(\cos\theta; \alpha) = \frac{1}{2\pi}\int_0^\pi \frac{(q, \alpha^2/\beta^2, \mathrm{e}^{2\mathrm{i}\psi}, \mathrm{e}^{-2\mathrm{i}\psi}; q)_\infty\, \mathcal{E}_q(\cos\psi; \beta)\, \mathrm{d}\psi}{(\mathrm{e}^{\mathrm{i}\theta+\mathrm{i}\psi}\alpha/\beta, \mathrm{e}^{\mathrm{i}\theta-\mathrm{i}\psi}\alpha/\beta, \mathrm{e}^{-\mathrm{i}\theta+\mathrm{i}\psi}\alpha/\beta, \mathrm{e}^{-\mathrm{i}\theta-\mathrm{i}\psi}\alpha/\beta; q)_\infty} \tag{13}$$

provided $\alpha < \beta$. Multiplying both sides of (13) by $\mathcal{E}_q(\cos\varphi;\beta)$ and then using the addition formula (8), the symmetry relation (7) and the definition (1) one obtains

$$\begin{aligned}
&\frac{(q\alpha^2;q^2)_\infty}{(q\beta^2;q^2)_\infty}\,\mathcal{E}_q(\cos\theta;\alpha)\,\mathcal{E}_q(\cos\varphi;\beta)\\
&\quad=\frac{1}{2\pi}\int_0^\pi \frac{(q,\alpha^2/\beta^2,\mathrm{e}^{2\mathrm{i}\psi},\mathrm{e}^{-2\mathrm{i}\psi};q)_\infty\,\mathcal{E}_q(\cos\varphi,\cos\psi;\beta)}{(\mathrm{e}^{\mathrm{i}\theta+\mathrm{i}\psi}\alpha/\beta,\mathrm{e}^{\mathrm{i}\theta-\mathrm{i}\psi}\alpha/\beta,\mathrm{e}^{-\mathrm{i}\theta+\mathrm{i}\psi}\alpha/\beta,\mathrm{e}^{-\mathrm{i}\theta-\mathrm{i}\psi}\alpha/\beta;q)_\infty}\,\mathrm{d}\psi\\
&\quad=\frac{(\beta^2;q^2)_\infty}{(q\beta^2;q^2)_\infty}(q,\alpha^2/\beta^2;q)_\infty\sum_{n=0}^\infty\frac{q^{n^2/4}}{(q;q)_n}(\beta\mathrm{e}^{-\mathrm{i}\varphi})^n\\
&\qquad\times\frac{1}{2\pi}\int_0^\pi\frac{(\mathrm{e}^{2\mathrm{i}\psi},\mathrm{e}^{-2\mathrm{i}\psi},-q^{(1-n)/2}\mathrm{e}^{\mathrm{i}\varphi+\mathrm{i}\psi},-q^{(1-n)/2}\mathrm{e}^{\mathrm{i}\varphi-\mathrm{i}\psi};q)_\infty}{(-q^{(1+n)/2}\mathrm{e}^{\mathrm{i}\varphi+\mathrm{i}\psi},-q^{(1+n)/2}\mathrm{e}^{\mathrm{i}\varphi-\mathrm{i}\psi};q)_\infty}\\
&\qquad\times\frac{\mathrm{d}\psi}{(\mathrm{e}^{\mathrm{i}\theta+\mathrm{i}\psi}\alpha/\beta,\mathrm{e}^{\mathrm{i}\theta-\mathrm{i}\psi}\alpha/\beta,\mathrm{e}^{-\mathrm{i}\theta+\mathrm{i}\psi}\alpha/\beta,\mathrm{e}^{-\mathrm{i}\theta-\mathrm{i}\psi}\alpha/\beta;q)_\infty}.
\end{aligned}\tag{14}$$

The last integral can be evaluated by the special case $a = b = 0$ of the Nassrallah and Rahman integral,

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^\pi\frac{(\mathrm{e}^{2\mathrm{i}\psi},\mathrm{e}^{-2\mathrm{i}\psi},g\mathrm{e}^{\mathrm{i}\psi},g\mathrm{e}^{-\mathrm{i}\psi};\,q)_\infty\,\mathrm{d}\psi}{(c\mathrm{e}^{\mathrm{i}\psi},c\mathrm{e}^{-\mathrm{i}\psi},d\mathrm{e}^{\mathrm{i}\psi},d\mathrm{e}^{-\mathrm{i}\psi},f\mathrm{e}^{\mathrm{i}\psi},f\mathrm{e}^{-\mathrm{i}\psi};q)_\infty}\\
&\quad=\frac{(cg;q)_\infty}{(q,cd,cf;q)_\infty}\,{}_2\varphi_1\left(\begin{matrix}g/d,g/f\\ cg\end{matrix};q,df\right)
\end{aligned}\tag{15}$$

(see (6.3.2) and (6.3.8) of [6]). Therefore,

$$\begin{aligned}
&\mathcal{E}_q(\cos\theta;\alpha)\,\mathcal{E}_q(\cos\varphi;\beta)\\
&\quad=\frac{(\beta^2;q^2)_\infty}{(q\alpha^2;q^2)_\infty}\sum_{n=0}^\infty\frac{q^{n^2/4}}{(q;q)_n}\,\beta^n\mathrm{e}^{-\mathrm{i}n\varphi}(-q^{(1-n)/2}\mathrm{e}^{\mathrm{i}\theta+\mathrm{i}\varphi}\alpha/\beta;q)_n\\
&\qquad\times{}_2\varphi_1\left(\begin{matrix}-q^{(1-n)/2}\mathrm{e}^{\mathrm{i}\theta+\mathrm{i}\varphi}\beta/\alpha,q^{-n}\\ -q^{(1-n)/2}\mathrm{e}^{\mathrm{i}\theta+\mathrm{i}\varphi}\alpha/\beta\end{matrix};q,-q^{(1+n)/2}\mathrm{e}^{\mathrm{i}\varphi-\mathrm{i}\theta}\alpha/\beta\right).
\end{aligned}\tag{16}$$

Use of the transformation (III.3) of [6] completes the proof. □

Changing the order of summation on the right-hand side of (11) one obtains an alternative form

$$\begin{aligned}
\mathcal{E}_q(x,y;\alpha,\beta)&=\frac{(\beta^2;q^2)_\infty}{(q\alpha^2;q^2)_\infty}\sum_{k=0}^\infty\frac{(\alpha^2/\beta^2;q)_k}{(q;q)_k}q^{k^2/4}(\beta\mathrm{e}^{\mathrm{i}\varphi})^k\\
&\quad\times\sum_{n=0}^\infty\frac{q^{n(n-2k)/4}\beta^n}{(q;q)_k}\mathrm{e}^{-\mathrm{i}n\varphi}(-q^{(1-n+k)/2}\mathrm{e}^{\mathrm{i}\theta+\mathrm{i}\varphi}\alpha/\beta,-q^{(1-n+k)/2}\mathrm{e}^{\mathrm{i}\varphi-\mathrm{i}\theta}\alpha/\beta;q)_n.
\end{aligned}\tag{17}$$

When $\beta = \alpha$ the first sum terminates and we obtain (1) once again. The second sum can be reduced to the sum of two ${}_4\varphi_3$-series; this expression is too lengthy to present here.

Function $\mathcal{E}_q(x, y;\alpha,\beta)$ on the right-hand side of (11) is an analogue of $\exp(\alpha x + \beta y)$. Indeed, from (16)

$$\begin{aligned}
&\lim_{q\to1^-}\mathcal{E}_q(x,y;(1-q)\alpha/2,(1-q)\beta/2)\\
&\quad=\sum_{n=0}^\infty\frac{(\beta/2)^n}{n!}\mathrm{e}^{-\mathrm{i}n\varphi}(1+\mathrm{e}^{\mathrm{i}\theta+\mathrm{i}\varphi}\alpha/\beta)^n
\end{aligned}$$

$$\times \sum_{k=0}^{n} \frac{(-n)_k}{k!} \frac{(1+\mathrm{e}^{\mathrm{i}\theta+\mathrm{i}\varphi}\beta/\alpha)^k(-\mathrm{e}^{\mathrm{i}\varphi-\mathrm{i}\theta}\alpha/\beta)^k}{(1+\mathrm{e}^{\mathrm{i}\theta+\mathrm{i}\varphi}\alpha/\beta)^k}$$

$$= \sum_{n=0}^{\infty} \frac{(1/2)^n}{n!}(\alpha\mathrm{e}^{\mathrm{i}\theta}+\beta\mathrm{e}^{-\mathrm{i}\varphi})^n\left(1+\frac{\alpha\mathrm{e}^{-\mathrm{i}\theta}+\beta\mathrm{e}^{\mathrm{i}\varphi}}{\alpha\mathrm{e}^{\mathrm{i}\theta}+\beta\mathrm{e}^{-\mathrm{i}\varphi}}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha x+\beta y)^n}{n!} = \exp(\alpha x+\beta y) \tag{18}$$

by the binomial theorem.

Function $u = \mathcal{E}_q(x, y; \alpha, \beta)$ in (11), (12) and (17) is a double-series solution of the difference equation

$$\frac{\delta}{\delta x}\,\mathcal{E}_q(x, y; \alpha, \beta) = \frac{2q^{1/4}\alpha}{1-q}\,\mathcal{E}_q(x, y; \alpha, \beta) \tag{19}$$

which is an analogue of

$$\frac{\mathrm{d}}{\mathrm{d}x}\exp(\alpha x+\beta y) = \alpha\exp(\alpha x+\beta y) \tag{20}$$

on a q-quadratic grid. Operator $\delta/\delta x$ is the standard Askey–Wilson divided difference operator

$$\frac{\delta u(z)}{\delta x(z)} = \frac{u(z+1/2)-u(z-1/2)}{x(z+1/2)-x(z-1/2)} \tag{21}$$

with $x = (q^z+q^{-z})/2 = \cos\theta$, $q^z = \mathrm{e}^{\mathrm{i}\theta}$. Applying this operator to (19) once again one obtains

$$\frac{\delta^2 u}{\delta x^2} = \left(\frac{2q^{1/4}\alpha}{1-q}\right)^2 u. \tag{22}$$

The method of solution of this equation discussed in [14] (see also references therein) does not involve the double-series solution found here.

Function $\mathcal{E}_q(x, y; \alpha, \beta)$ satisfies the following simple properties:

$$\mathcal{E}_q(x, y; \alpha, \alpha) = \mathcal{E}_q(x, y; \alpha) \qquad \mathcal{E}_q(x, y; \alpha, \beta) = \mathcal{E}_q(y, x; \beta, \alpha) \tag{23}$$

$$\mathcal{E}_q(x, 0; \alpha, \beta) = \mathcal{E}_q(x; \alpha) \qquad \mathcal{E}_q(0, y; \alpha, \beta) = \mathcal{E}_q(y; \beta). \tag{24}$$

Equation (12) leads also to the product formula

$$\mathcal{E}_q(x, y; \alpha, \beta)\,\mathcal{E}_q(z, w; \gamma, \delta) = \mathcal{E}_q(x, z; \alpha, \gamma)\,\mathcal{E}_q(y, w; \beta, \delta) \tag{25}$$

which is, obviously, a q-analogue of

$$\exp(\alpha x+\beta y)\exp(\gamma z+\delta w) = \exp(\alpha x+\gamma z)\exp(\beta y+\delta w).$$

Equation (25) is an extension of the product formula (7.7) of [14] in the case of the q-quadratic lattice under consideration.

Two limiting cases of (16) are of interest. When $\beta \to 0$, we obtain the generating function for the continuous q-Hermite polynomials

$$(q\alpha^2; q^2)_\infty \mathcal{E}_q(x; \alpha) = \sum_{n=0}^{\infty} \frac{q^{n^2/4}}{(q; q)_n}\alpha^n\, H_n(x|q) \tag{26}$$

discussed in [2, 4, 9, 11, 12] and [14]. One needs this generating function in order to derive the connecting formula (13) (see [1]). Another limiting case, $\alpha \to 0$, results in the following generating relation found in [11]:

$$\mathcal{E}_q(x; \beta) = (\beta^2; q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2/4}}{(q; q)_n}\beta^n H_n(x|q^{-1}) \tag{27}$$

where

$$H_n(\cos\theta|q^{-1}) = \sum_{k=0}^{n} q^{k^2-kn} \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} e^{i(n-2k)\theta}. \tag{28}$$

One can also introduce basic trigonometric functions

$$C_q(x, y; \omega, \varkappa) = \tfrac{1}{2}(\mathcal{E}_q(x, y; i\omega, i\varkappa) + \mathcal{E}_q(x, y; -i\omega, -i\varkappa)) \tag{29}$$

$$S_q(x, y; \omega, \varkappa) = \frac{1}{2i}(\mathcal{E}_q(x, y; i\omega, i\varkappa) - \mathcal{E}_q(x, y; -i\omega, -i\varkappa)) \tag{30}$$

as analogues of $\cos(\omega x + \varkappa y)$ and $\sin(\omega x + \varkappa y)$, respectively. The following addition formulae hold.

Theorem 3.

$$C_q(x, y; \omega, \varkappa) = C_q(x; \omega)C_q(y; \varkappa) - S_q(x; \omega)S_q(y; \varkappa) \tag{31}$$

$$S_q(x, y; \omega, \varkappa) = S_q(x; \omega)C_q(y; \varkappa) + C_q(x; \omega)S_q(y; \varkappa). \tag{32}$$

These formulae are, obviously, q-analogues of

$$\cos(\omega x + \varkappa y) = \cos\omega x \cos\varkappa y - \sin\omega x \sin\varkappa y \tag{33}$$

$$\sin(\omega x + \varkappa y) = \sin\omega x \cos\varkappa y + \cos\omega x \sin\varkappa y. \tag{34}$$

The special case $\varkappa = \omega$ of formulae (31) and (32) was discussed in [14].

Addition formula (11) deserves further investigation. For example, it is worth understanding the similarity of the function $\mathcal{E}_q(x, y; \alpha, \beta)$ introduced here to the generating functions for the continuous q-ultraspherical polynomials and the continuous q-Jacobi polynomials found in [8]. A group-theoretical interpretation of the q-addition theorems is another interesting question under consideration.

The author thanks Joaquin Bustoz for valuable discussion. He was supported by NSF grant DMS 9803443.

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